1 Background & Notation

We work in the three-dimensional Euclidean vector space $\mathbb{E}^3$. Let $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a given (fixed) basis which is orthonormal with respect to the given inner-product $\langle \cdot, \cdot \rangle : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}$ on $\mathbb{E}^3$. (i.e. $\mathcal{A}$ is a basis for $\mathbb{E}^3$, and $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$ $\Rightarrow \|\mathbf{e}_i\| = \langle \mathbf{e}_i, \mathbf{e}_i \rangle^{\frac{1}{2}} = 1$ for $i = 1, 2, 3$).

Given $\mathbf{x} \in \mathbb{E}^3$ we can write $\mathbf{x} = \sum_{i=1}^3 x_i \mathbf{e}_i$, where $x_j = \langle \mathbf{x}, \mathbf{e}_j \rangle = \langle \sum_{i=1}^3 x_i \mathbf{e}_i, \mathbf{e}_j \rangle = \sum_{i=1}^3 \mathbf{x}_i \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \mathbf{x}_j$, for $j = 1, 2, 3$ and x_1, x_2, x_3 are called the *components* of the vector $\mathbf{x}$ in the basis $\mathcal{A}$.

Given $\mathbf{x} = \sum_{i=1}^3 x_i \mathbf{e}_i,\ \mathbf{y} = \sum_{i=1}^3 y_j \mathbf{e}_j \in \mathbb{E}^3$, then the inner product of $\mathbf{x}$ and $\mathbf{y}$ is:

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \langle \sum_{i=1}^3 \mathbf{x}_i \mathbf{e}_i, \sum_{j=1}^3 \mathbf{y}_j \mathbf{e}_j \rangle \\
&= \sum_{i=1}^3 \mathbf{x}_i \langle \mathbf{e}_i, \sum_{j=1}^3 \mathbf{y}_j \mathbf{e}_j \rangle \\
&= \sum_{i=1}^3 \mathbf{x}_i \left(\sum_{j=1}^3 \mathbf{y}_j \langle \mathbf{e}_i, \mathbf{e}_j \rangle \right) \\
&= \sum_{i=1}^3 \mathbf{x}_i \mathbf{y}_i
\end{aligned}$$

Note: We have used linearity of the inner product, and the fact that $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$.

Hence the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ of $\mathbf{x}, \mathbf{y} \in \mathbb{E}^3$ is equal to the standard dot product of the two vectors of components $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$.

Moreover, the induced norm on $\mathbb{E}^3$ is given by $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \big((x_1)^2 + (x_2)^2 + (x_3)^2 \big)$, which is equal to the standard norm on $\mathbb{R}^3$ induced by the dot product, on the vectors of components in $\mathbb{R}^3$.

If $\mathbf{A} = (a_{ij}), \mathbf{B} = (b_{ij}) \in M^{n \times n}$, then the standard inner-product is given by

$$\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{tr}\left(\mathbf{A}^T \mathbf{B} \right) = a_{ij} b_{ij},$$

where tr denotes the trace. The induced norm is

$$|\mathbf{A}| = \langle \mathbf{A}, \mathbf{A} \rangle^{\frac{1}{2}} = (a_{ij} a_{ij})^{\frac{1}{2}}.$$

(This inner-product on $M^{n \times n}$ is the same as that obtained by identifying $M^{n \times n}$ with $\mathbb{R}^{n^2}$ and then taking the dot product on $\mathbb{R}^{n^2}$.)

Terminology: A *cartesian frame* is a choice of origin together with a right-handed orthonormal basis/coordinate system.

Remark 1.1. *From now on, we identify $\mathbb{E}^3, < \cdot, \cdot >$ with given cartesian basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with the space $\mathbb{R}^3$ endowed with the standard dot product (in which identifying $\mathbf{x} = x_i \mathbf{e}_i \in \mathbb{E}^3$ with $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$) and standard basis $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,*

$\hat{e}_3 = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. *Later in this course, when we study Cartesian Tensors we will investigate how the models we develop depend on the particular choice of Cartesian basis. A key principal in the modelling of continuous media (e.g. fluids, solids) is that the form and structure of property formulated equations describing physical systems, and the relationship between physical quantities, should not depend on the choice of the cartesian coordinate system of the observer.*

1.1 Einstein Summation Convention

Under this useful convention, in any expression in which an index appears twice in a product, the expression is assumed to be summed over all possible values of the index. The index of summation is called a *dummy* index.

Convention Rules

1. We sum over any index appearing twice in a product.
2. No index may appear more than twice in a term.
3. Any index which only appears once in a single term must occur exactly once in a every term of a sum and it is called a *free* index.

Examples

(i) If $\mathbf{x} \in \mathbb{E}^3, \mathbf{x} = \sum_{i=1}^{3} \mathbf{x}_i \mathbf{e}_i$, then we write that as

$$\mathbf{x} = x_i \mathbf{e}_i = x_k \mathbf{e}_k \qquad (i \text{ and } k \text{ are } \textit{dummy} \text{ indices}).$$

(ii) Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_k b_k \quad (k \text{ is a } \textit{dummy} \text{ index}).$$

(iii) Let $A = (a_{ij}) \in M^{3\times 3}, \mathbf{x} = (x_i), \mathbf{b} = (b_i), \mathbf{z} = (z_i) \in \mathbb{R}^3$ and take any scalar $\alpha \in \mathbb{R}$. Then the system of equations $A\mathbf{x} = \mathbf{b} + \alpha\mathbf{z}$ (i.e. $\sum_{j=1}^{3} a_{ij} x_j = b_i + \alpha z_i$, for $i = 1, 2, 3$) is represented by

$$a_{ij} x_j = b_i + \alpha z_i,$$

where j is a dummy index and i is a free index.

(iv) Let $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij}) \in M^{3\times 3}$. The system of equations

$$d_{ij} = a_{ik} b_{kl} c_{lj}$$

where k, l are dummy indices and i, j are free indices, represents the matrix equation

$$D = ABC.$$

(v) Let $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}) \in M^{3\times 3}$. Then the matrix equation $AC^T = B$ is equivalent to $a_{ik} c_{jk} = b_{ij}$, where k is a dummy index and i, j are free indices.

Gradient

The gradient of a scalar function $f(\mathbf{x})$ is the vector field

$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i.$$

1.2 Useful examples, Vector Identities and Notation

Kronecker Delta

This is denoted δ_{ij} and defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Alternating (or Permutation) Symbol

This is defined by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ is an even permutation of } 1,2,3 \\ -1 & \text{if } i,j,k \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{otherwise.} \end{cases}$$

for $i,j,k = 1,2,3$.

Even permutations: 1,2,3, 2,3,1 and 3,1,2 .

Odd permutations: 1,3,2, 3,2,1 and 2,1,3

(Note: $\epsilon_{ijk} = 0$ if any two of the indices i,j,k are equal.)

Vector Product

The vector product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{E}^3$ (with components x_1, x_2, x_3 and y_1, y_2, y_3 respectively) in the right-handed orthonormal basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined by

$$\mathbf{x} \times \mathbf{y} = \epsilon_{ijk}\, x_j\, y_k\, \mathbf{e}_i.$$

The vector product can be expressed as the formal determinant

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2)\mathbf{e}_1 - (x_1y_3 - x_3y_1)\mathbf{e}_2 + (x_1y_2 - x_2y_1)\mathbf{e}_3.$$

Geometrically, $\mathbf{x} \times \mathbf{y}$ is orthogonal to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$ (with $\mathbf{x}, \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ oriented in a right-handed sense). Moreover, $||\mathbf{x} \times \mathbf{y}|| = ||\mathbf{x}||\, ||\mathbf{y}|| \sin\theta$ which is the area of the parallelogram with sides parallel to $\mathbf{x}$ and $\mathbf{y}$, and θ is the angle between the vectors.

Remark. If $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a right-handed orthonormal basis, then $\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k$, i.e., $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$, etc.

Curl

The curl of a vector field $\mathbf{g}(\mathbf{x}) = (g_i(\mathbf{x}))$ is the vector field

$$(\nabla \times \mathbf{g}) = \epsilon_{ijk} \frac{\partial g_k}{\partial x_j} \mathbf{e}_i$$

and is formally given by the determinant

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ g_1 & g_2 & g_3 \end{vmatrix}.$$

Determinant.

The determinant of $A = (a_{ij}) \in M^{3\times 3}$ is given by

$$\det A = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} = \epsilon_{ijk} a_{i1} a_{j2} a_{k3} = \det(A^T).$$

More generally,

$$\epsilon_{ijk} a_{pi} a_{qj} a_{rk} = \epsilon_{pqr} \det A,$$

and hence

$$\epsilon_{pqr}\epsilon_{ijk} a_{pi} a_{qj} a_{rk} = \epsilon_{pqr}\epsilon_{pqr} \det A \;\Rightarrow\; \det A = \frac{1}{6}\epsilon_{pqr}\epsilon_{ijk} a_{pi} a_{qj} a_{rk}.$$

The Scalar Triple Product of $\mathbf{X} = X_i \mathbf{e}_i, \mathbf{Y} = Y_j \mathbf{e}_j, \mathbf{Z} = Z_k \mathbf{e}_k \in \mathbb{E}^3$ is defined by

$$\langle \mathbf{X}, \mathbf{Y} \times \mathbf{Z} \rangle = \epsilon_{ijk} X_i Y_j Z_k = \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix}.$$

Clearly (e.g, by properties of the determinant)

$$\langle \mathbf{X}, \mathbf{Y} \times \mathbf{Z} \rangle = \langle \mathbf{Y}, \mathbf{Z} \times \mathbf{X} \rangle = \langle \mathbf{Z}, \mathbf{X} \times \mathbf{Y} \rangle \quad .$$

Geometrically, the absolute value of the triple scalar product is the volume of the 3d parallelepiped with edges parallel to the vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

Remark. A very useful identity linking the Kronecker delta and alternating symbol is

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} \quad .$$